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PROBLEM OF A STRONG EXPLOSION IN A WEAKLY COMPRESSIBLE MEDIUM

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The problem of one-dimensional motions of an inviscid non-heat-conducting polytropic gas, which arises in the description of endothermal combustion-type physical processes, is considered. The medium is assumed to be weakly compressible (the adiabatic exponent is much greater than unity). The gas motion is initiated by a point (explosive) energy release. The goal of the paper is to derive approximate equations that describe the dynamics of strong explosions.

1. Formulation of the Problem. We shall assume that phase transformations of a medium occur according to a shock wave–phase transition pattern. The phase transition occurs if the pressure behind the shock wave is $p \geqslant p_*$ (p_* is a certain critical pressure). The continuity conditions for the flow, momentum, and energy of the substance should be satisfied on the surface of a discontinuity, which serves as the phase-transition zone, with allowance for the energy spent for the occurrence of the phase transition.

We shall consider one-dimensional motions with plane, cylindrical, or spherical waves. Let a gas with the adiabatic exponent γ_0 occupy the region $r \geqslant 0$ at the initial moment $t = 0$. The energy E_0 is released at the coordinate origin $r = 0$ at the initial moment. After that, a strong explosion whose position is described by the equality $r = R(t)$ moves to the right. The region $0 < r < R(t)$ is occupied by the moving gas with the adiabatic exponent γ. The differential equations which describe the continuous motion of the gas for $0 < r < R(t)$ are of the form [1]

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu \rho u) = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \quad \frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + u\frac{\partial}{\partial r}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{1.1}$$

where ρ, p, and u are the density, pressure, and velocity of the gas, respectively. The first, second, and last of these equations correspond to the law of conservation of mass, momentum, and entropy in the polytropic gas particles, respectively. The geometry parameter $\nu = 0$ refers to plane waves, $\nu = 1$ to cylindrical flows, and $\nu = 2$ to spherical waves.

At the discontinuity $r = R(t)$, the following equalities should be satisfied:

$$\begin{gathered} \rho(R' - u) = \rho_0 R', \quad \rho(R' - u)^2 + p = \rho_0 R'^2 + p_0, \\ \frac{1}{2}(R' - u)^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{\gamma_0}{\gamma_0 - 1}\frac{p_0}{\rho_0} + \frac{1}{2}R'^2 - e_0. \end{gathered} \tag{1.2}$$

Here p_0 and ρ_0 are the pressure and density in the gas at rest, and e_0 is the specific energy consumed during the phase transition.

For the total energy, the following formulas hold:

$$\int_0^R r^\nu \left(\frac{\rho u^2}{2} + \frac{p}{\gamma - 1} - \frac{p_0}{\gamma_0 - 1} + \rho_0 e_0\right) dr = S_\nu E_0, \tag{1.3}$$

where $S_\nu = 1/2$, $1/2\pi$, and $1/4\pi$ for $\nu = 0$, 1, and 2, respectively.

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Thus, we should find the function $R(t)$ and also the density ρ, the pressure p, and the velocity u, which were determined for $0 < r < R(t)$ from Eqs. (1.1) with boundary conditions (1.2) at a strong discontinuity for $r = R(t)$, and the condition

$$u = 0 \tag{1.4}$$

on the axis of symmetry with $r = 0$.

At the initial moment of time, for $t = 0$ we have

$$R(0) = 0. \tag{1.5}$$

In addition, the law of conservation of energy (1.3) should be fulfilled for all times $t > 0$.

Remark 1. If one admits the existence of a free boundary $r = R_0(t) < R(t)$ such that there is no gas (vacuum) inside a "sphere" $r < R_0(t)$, the problem will be formulated differently: find the functions $R_0(t)$ and $R(t)$ and also the density ρ, the pressure p, and the velocity u, which were determined for $R_0(t) < r < R(t)$ from system (1.1) with boundary conditions (1.2) at a strong discontinuity for $r = R(t)$ and the condition

$$u = R_0'(t), \quad p = 0 \tag{1.6}$$

at $r = R_0(t)$. At the initial moment of time, for $t = 0$, we have

$$R_0(0) = 0, \quad R(0) = 0. \tag{1.7}$$

In addition, the law of conservation of energy

$$\int_{R_0}^{R} r^\nu \left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}\right) dr + \frac{R^{\nu+1}}{\nu + 1}\left(\rho_0 e_0 - \frac{p_0}{\gamma_0 - 1}\right) = S_\nu E_0 \tag{1.8}$$

should be satisfied for all times $t > 0$.

Remark 2. Equalities (1.2) for $e_0 = 0$ and $\gamma = \gamma_0$ relate four unknown quantities, namely, p, ρ, u, and R'. Therefore, one can find $p = \Phi(R')$ for a shock wave. It follows from the phase-transition law that the solution of the formulated problem has a sense for times at which the inequality $\Phi(R') \geqslant p_*$ is fulfilled.

2. Assumptions of the Medium's Weak Compressibility and a Strong Explosion. We assume that $\varepsilon = 1/\sqrt{\gamma} \ll 1$ and $\gamma_0 = \gamma/\alpha$. Let L be the characteristic length. We set

$$\bar{r} = \frac{r}{L}, \quad \bar{t} = \frac{t}{\varepsilon L}\sqrt{\frac{p_0}{\rho_0}}, \quad p = p_0\bar{p}, \quad u = \varepsilon\sqrt{\frac{p_0}{\rho_0}}\,\bar{u}, \quad R = L\bar{R}(\bar{t}),$$

$$\rho = \rho_0(1 + \varepsilon^2\theta), \quad S_\nu E_0 = \varepsilon^2 p_0 L^{\nu+1} E, \quad e_0 = \varepsilon^2\frac{p_0}{\rho_0}(e + \alpha), \quad p_* = p_0\bar{p}_*.$$

If there is a free surface, we also assume that $R_0 = \varepsilon^2 L\bar{R}_0(\bar{t})$.

In new variables (the bar is omitted), with accuracy of up to ε-junior terms, Eqs. (1.1) take the form

$$\frac{\partial}{\partial t}(r^\nu\theta) + \frac{\partial}{\partial r}(r^\nu u) = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} = 0, \quad \frac{\partial}{\partial t}(pe^\theta) = 0 \qquad (0 < r < R(t)). \tag{2.1}$$

It follows from the last relation that

$$p = f(r)e^{-\theta} \tag{2.2}$$

with a certain negative function $f(r)$.

The boundary conditions (1.2) will be approximately satisfied if

$$R'\theta = u, \quad \frac{1}{2}u^2 + (1 - \theta)p + e = 0 \qquad (r = R(t)); \tag{2.3}$$

$$R'u = p - 1 \qquad (r = R(t)). \tag{2.4}$$

Under these assumptions, the laws of conservation of energy (1.3) and (1.8) take the form

$$\int_0^R r^\nu \left(\frac{1}{2}u^2 + p\right) dr + \frac{e}{\nu+1} R^{\nu+1} = E. \tag{2.5}$$

Remark 3. In subsequent calculations, we assume that $e \geqslant 0$.

Let $E \gg 1$ and $e \gg 1$. After that, at least for small times, we have $p \gg 1$ behind the strong-discontinuity front, and hence the unity on the right-hand side of equality (2.4) can be ignored:

$$R' u = p \qquad (r = R(t)). \tag{2.6}$$

In the initial variables, this implies that the pressure behind the strong-discontinuity front is considerably higher than that in the gas at rest. Excluding the function θ from the boundary conditions (2.3) by means of relation (2.6), we obtain

$$(1/2)\, u^2 = p + e \qquad (r = R(t)). \tag{2.7}$$

After the function θ is discarded from Eqs. (2.1) and (2.2), we have the system

$$\frac{\partial}{\partial t}(r^\nu \ln p) + \frac{\partial}{\partial r}(r^\nu u) = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} = 0 \quad (0 < r < R(t)). \tag{2.8}$$

Thus, under the assumptions of the medium's weak compressibility and of a strong explosion, problem (1.1)–(1.5) [or (1.1)–(1.7)] has reduced to the search for the pressure p, the velocity u, and the position of the strong-discontinuity front $R(t)$ from system (2.8) with boundary conditions in the strong-discontinuity front (2.6) and (2.7) and the condition on the axis of symmetry

$$u = 0 \quad \text{or} \quad p = 0 \qquad (r = 0) \tag{2.9}$$

and the initial condition

$$R(0) = 0.$$

In addition, the law of conservation of energy (2.5) should be satisfied for all times $t > 0$.

Remark 4. The phase-transition condition is of the form

$$R'^2 \geqslant (1/2)\, p_*$$

within the framework of the reduced approximation.

3. Self-Modeling Solutions ($e = 0$). As in [1], we shall search for the position of a shock wave and the velocity and the pressure in the form

$$R(t) = \left[\frac{1}{4} a(\nu+3)^2\right]^{1/(\nu+3)} t^{2/(\nu+3)}; \tag{3.1}$$

$$u(r,t) = R'(t) x V(x), \quad p(r,t) = R^{-\nu-1}(t) P(x), \tag{3.2}$$

where $x = r/R(t)$. In accordance with Eqs. (2.8), the functions V and P are found from the system

$$(x^{\nu+1}P(x))' = P(x)(x^{\nu+1}V(x))' \qquad (0 < x < 1); \tag{3.3}$$

$$a x^{-(\nu-1)/2}(x^{(\nu+3)/2}V(x))' = P'(x) \qquad (0 < x < 1) \tag{3.4}$$

with boundary conditions

$$aV(1) = P(1), \quad (1/2) a V^2(1) = P(1),$$

which are a consequence of equalities (2.6) and (2.7) for $e = 0$. Thus,

$$V(1) = 2, \quad P(1) = 2a. \tag{3.5}$$

System (3.3) and (3.4) permits a decrease in the order, because the Sedov integral is valid for it:

$$x^{\nu+1}\left[\frac{1}{2}ax^2V^2+P\right]-x^{\nu+1}PV=C.$$

For $x=0$, it follows from the condition on the axis of symmetry that $C=0$, and therefore

$$P=\frac{a}{2}\frac{x^2V^2}{V-1}. \tag{3.6}$$

We note that $P=2a$ for $x=1$ and $V=2$, and hence the second equality in (3.5) is a consequence of the first equality.

The constant a is determined from the law of conservation of energy (2.5):

$$\frac{a}{2}\int_0^1\frac{x^{\nu+2}V^3}{V-1}\,dx=E. \tag{3.7}$$

Excluding the pressure from Eq. (3.4) with the use of equality (3.6), we find an equation on V:

$$x(V^2-2V+2)V'=-(\nu+1)V(V-1)\left(V-\frac{\nu+3}{\nu+1}\right).$$

It is easy to integrate this equation:

$$V^{\nu_1}|V-1|^{-\nu_2}\left|V-\frac{\nu+3}{\nu+1}\right|^{\nu_3}=\frac{C}{x^{\nu+1}},$$

where

$$\nu_1=2\frac{\nu+1}{\nu+3},\quad \nu_2=\frac{\nu+1}{2},\quad \nu_3=\frac{\nu^2+2\nu+5}{2(\nu+3)}.$$

For the boundary conditions (3.5), it follows that, for $\nu=2$, no self-modeling solutions with the finite energy integral (3.7) exist in the case of spherical symmetry.

For axisymmetrical flows ($\nu=1$), we have

$$V=2,\quad P=2ax^2,\quad a=E,\quad R(t)=(4E)^{1/4}t^{1/2}. \tag{3.8}$$

Finally, in the case of plane waves ($\nu=0$) we have $C=2^{2/3}$, $V(0)=1$, and

$$a=\frac{E}{2}\int_1^2\frac{V^2-2V+2}{(V-1)^{1/2}(3-V)^{7/2}}\,dV=\frac{3}{4}E,\quad R(t)=3\cdot 2^{-4/3}E^{1/3}t^{2/3}\approx 1.19055E^{1/3}t^{2/3}. \tag{3.9}$$

Remark 5. For plane waves, at $x\to 0$ we have

$$V\sim 1+2^{1/3}x^2,\quad P\sim a/2^{4/3},$$

and hence the free surface does not exist.

For cylindrical waves, the pressure and the velocity vanish on the axis of symmetry. The problem of the formation of a free surface has no solution within the framework of this approximation.

The absence of self-modeling solutions in the case of spherical symmetry means that the approximation of the medium's weak compressibility does not adequately describe the flow with a free surface.

4. Variational Principle.

Theorem. *The solutions of the equations of motion of a continuous medium* (2.6)–(2.9) *coincide with the extremals of the action functional*

$$\int_0^T\int_0^R\left(\frac{1}{2}u^2-p-e\right)r^\nu\,dr\,dt \tag{4.1}$$

under the additional condition

$$(r^\nu\ln p)_t+(r^\nu u)_r=0. \tag{4.2}$$

Proof. For any function $g(r,t)$, we set $\bar{g}(t) = g(R(t),t)$. The variation of the functional (4.1) relative to R immediately yields

$$\int_0^T \delta R\left(\frac{1}{2}\bar{u}^2 - \bar{p} - e\right) R^\nu \, dt.$$

The boundary condition (2.7) follows owing to the arbitrariness of δR. After that, we set

$$p = \exp\{r^{-\nu} f_r(r,t)\}, \quad u = -r^{-\nu} f_t(r,t) \tag{4.3}$$

with a smooth function f such that $r^{-\nu} f \to 0$ (or $r^{-\nu} f \to -\infty$) at $r \to 0$. Thereby the condition on the axis of symmetry (2.9) and the law of conservation of mass (4.2) for the functions u and p are satisfied automatically. In view of this, one can consider the functional

$$\int_0^T \int_0^R \left(\frac{1}{2} r^{-2\nu} f_t^2 - \exp\{r^{-\nu} f_r\} - e\right) r^\nu \, dr \, dt \tag{4.4}$$

instead of the functional (4.1) with restrictions (2.9) and (4.2). Varying this functional with respect to f under the assumption that $\delta f(r,0) = \delta f(r,T) = 0$, we obtain

$$\int_0^T \int_0^R (u_t + p_r)\, \delta f \, dr \, dt + \int_0^T (\bar{u}R' - \bar{p})\, \overline{\delta f} \, dt = 0 \tag{4.5}$$

after integration by parts. By virtue of the arbitrariness of δf and $\overline{\delta f}$, the second equation follows from (2.8) and the boundary condition (2.6). The theorem is proved.

5. Approximate Solutions. We shall search for approximate solutions of problem (2.6)–(2.9) as the extremals of the functional (4.1) in the classes of functions of the form (3.2) with the desired function $R(t)$ under the condition that the functions V and P satisfy relation (3.3).

For specified functions V and P, the functional (4.1) takes the form

$$S(R) = \int_0^T \left(\frac{1}{2} A(V) R'^2 R^{\nu+1} - \frac{e}{\nu+1} R^{\nu+1}\right) dt - TB(P), \tag{5.1}$$

where

$$A(V) = \frac{1}{2}\int_0^1 x^{\nu+2} V^2(x)\, dx, \quad B(P) = \int_0^1 x^\nu P(x)\, dx. \tag{5.2}$$

The integrand $F(R, R')$ on the right-hand side of (5.1) is not time-dependent explicitly. Therefore, the Euler equations of the functional $S(R)$ have the following first integral [2]: $-R' F_{R'} + F = C$. It is easy to verify that, with the constant C chosen properly, the first integral coincides with the energy integral (2.5) for functions u and p of the form (3.2):

$$AR'^2 R^{\nu+1} + B + \frac{e}{\nu+1} R^{\nu+1} = E. \tag{5.3}$$

We note that the equalities

$$-AR'' R^{\nu+2} = (\nu+1) AR'^2 R^{\nu+1} + eR^{\nu+1},$$

$$\frac{1}{T}\int_0^T R'^2 R^{\nu+1}\, dt = \frac{1}{A}\left(E - B - \frac{e}{\nu+1}\frac{1}{T}\int_0^T R^{\nu+1}\, dt\right)$$

follow from (5.3) with the use of differentiation and integration operations. These equalities can be reduced to the form

$$\frac{1}{T}\int\limits_0^T R'^2 R^{\nu+1}\,dt = a - b, \quad -\frac{1}{T}\int\limits_0^T R'' R^{\nu+2}\,dt = \frac{\nu+1}{2}\,a, \tag{5.4}$$

where

$$a = \frac{1}{A}(E - B), \quad b = \frac{e}{(\nu+1)A}\,\frac{1}{T}\int\limits_0^T R^{\nu+1}\,dt. \tag{5.5}$$

We set

$$f(r,t) = -r^{\nu+1}\ln R + R^{\nu+1}g(x) \qquad (x = r/R). \tag{5.6}$$

The equalities (3.2) for u and p with the functions

$$P(x) = \exp\{x^{-\nu}g'(x)\}, \quad V(x) = 1 - (\nu+1)x^{-\nu-1}g(x) + x^{-\nu}g'(x) \tag{5.7}$$

follow from the representations (4.3). With this choice of the functions V and P, relation (3.3) is satisfied automatically and, as a consequence, equality (4.2) holds.

Next, we shall search for the extremal of the functional (4.4) in the classes of functions of the form (5.6) for a specified function $R(t)$. The set of functions of the form (5.6) contains no time-finite functions. Therefore, we shall consider the sequence of variations of the form

$$\delta f_n = \varphi_n(t)R^{\nu+1}\,\delta g(x),$$

where $\varphi_n(t)$ are uniformly bounded smooth nonnegative functions. They vanish for $t = 0$ and $t = T$ and converge to unity for $n \to \infty$ inside the interval $(0,T)$. Substituting δf_n into formula (4.5) within the limit for $n \to \infty$ after the replacement of r by x, we obtain

$$ax^{-(\nu-1)/2}\Big(x^{(\nu+3)/2}V\Big)' - bx(xV)' = P' \qquad (0 < x < 1); \tag{5.8}$$

$$P(1) = (a - b)V(1) \tag{5.9}$$

owing to the arbitrariness of $\delta g(x)$ and $\delta g(1)$.

Based on the aforesaid, one can formulate the problem of the search for approximate solutions.

Approximate Formulation I. One should find the function $R(t)$ on the interval $(0,T)$, the functions $V(x)$ and $P(x)$ on the interval (0,1), and also the parameters a and b from the energy integral (5.3), Eqs. (3.3) and (5.8), relations (5.5), the boundary condition (5.9), and the initial condition $R(0) = 0$. For $x = 0$, one of the two conditions

$$\lim_{x\to 0} xV(x) = 0 \quad \text{or} \quad P(0) = 0$$

should be satisfied. In the first case, there is no free surface. In the case of the second boundary condition, the free surface exists if $\lim\limits_{x\to 0} xV(x) > 0$.

Remark 6. For $e = 0$, the approximate formulation coincides with the problem of the search for self-modeling solutions.

We obtain a simpler model if we take the family of functions $g(x,\xi)$, which depends on the vector parameter $\xi = (\xi_1,\ldots,\xi_k)$, in equality (5.6). In this case, the functions P and V, which were defined by formulas (5.7), are known functions of the variable x and the parameter ξ which satisfy relation (3.3). The quantities A and B determined in (5.2) are known functions of ξ. As before, the derivatives with respect to x are primed.

We shall find the extremal of the functional (4.4) in the class of functions (5.6) with functions $g(x,\xi)$, which belong to the family described above.

The variation of the functional (4.4) relative to R produces, as before, the energy integral (5.3).

To derive equations that determine the parameter ξ, we consider the sequence of perturbations of the function f of the form

$$\delta f_n = \varphi_n(t)\psi_n(x)\,\delta\xi\,\nabla_\xi g(x,\xi)R^{\nu+1}(t).$$

The sequences $\varphi_n(t)$ and $\psi_n(x)$ consist of uniformly bounded nonnegative functions; we have $\varphi_n(t) = 0$ for $t = 0$ and $t = T$, and we have $\psi_n(x) = 0$ for $x = 0$ and $x = 2$. In addition, we have $\varphi_n(t) \to 1$ and $\psi_n(x) \to 1$ for $n \to \infty$. Passing to the limit relative to n in equality (4.5) and with r replaced by x, we obtain

$$\frac{\nu+1}{2}a\int_0^1 xV\nabla_\xi g\,dx + (a-b)\int_0^1 x(xV)'\nabla_\xi g\,dx = \int_0^1 P'\nabla_\xi g\,dx + [(a-b)V(1,\xi) - P(1,\xi)]\nabla_\xi g(1,\xi)$$

owing to the arbitrariness of $\delta\xi$. Integrating by parts and excluding the function g by means of formulas (5.7), we obtain

$$\left(\frac{\nu+3}{2}a - (\nu+2)b\right)\int_0^1 x^{\nu+2}VP^{-1}\nabla_\xi P\,dx$$

$$+\left(\frac{\nu-1}{2}a + b\right)\int_0^1 x^{\nu+2}V\nabla_\xi V\,dx = (\nu+1)\int_0^1 x^\nu\nabla_\xi P\,dx. \tag{5.10}$$

Approximate Formulation II. Let there be the set of functions $V(x,\xi)$ and $P(x,\xi)$ which depends on the vector parameter ξ. These functions satisfy the equality (3.3) and one of the following boundary conditions:

$$\lim_{x\to 0} xV(x,\xi) = 0 \quad \text{or} \quad P(0,\xi) = 0. \tag{5.11}$$

It is necessary to find the function $R(t)$ on the interval $(0,T)$ and the parameters a, b, and ξ from the energy integral (5.3), systems (5.10) and equalities (5.4), and the initial condition $R(0) = 0$.

Next, we assume that Eqs. (5.10) and the first relation in (5.4) determine the desired quantities $\xi(b)$ and $a(b)$ versus the parameter b. Formulas (5.2) define the functions $A(b)$ and $B(b)$. The problem of the search for the position of a strong discontinuity is reduced to finding the parameter b and the function $R(t)$ from the energy integral (5.3), the second equality in (5.4), and the initial condition $R(0) = 0$.

To solve Eq. (5.3) for $e \neq 0$, it is expedient to set

$$M(b) = \{e^{-1}(\nu+1)[E-B]\}^{1/(\nu+1)}, \quad K^2(b) = A^{-1}M^{-\nu-3}(E-B)$$

and to introduce the new desired function $Z = M^{-1}R$. The equation on $Z(t,b)$

$$Z'Z^{(\nu+1)/2} = K(1-Z^{\nu+1})^{1/2}$$

with zero initial data for $t = 0$ determines the dependence of t on Z and b:

$$t = K^{-1}F_\nu(Z), \tag{5.12}$$

where

$$F_\nu(Z) = \int_0^Z Z^{(\nu+1)/2}(1-Z^{\nu+1})^{-1/2}\,dZ. \tag{5.13}$$

The equality (5.12), which is treated as an equation on Z, determines $Z_*(b) = Z(T,b)$. In accordance with the second equation in (5.4), the parameter b should be found from the equation

$$b = T^{-1}(E-B)K^{-1}G_\nu(Z_*), \tag{5.14}$$

where

$$G_\nu(Z) = \int_0^Z Z^{3(\nu+1)/2}(1-Z^{\nu+1})^{-1/2}\,dZ. \tag{5.15}$$

Thereby the problem is completely defined. It was reduced to the solution of Eq. (5.14). We note that the integrals (5.13) and (5.15) are calculated via elementary functions for $\nu = 0$ and 1. For example, for $\nu = 1$, we have

$$F_1(Z) = 1 - (1 - Z^2)^{1/2}, \quad G_1(Z) = \frac{2}{3} - \frac{1}{3}(2 + Z^2)\sqrt{1 - Z^2}.$$

According to Remark 4, it is natural to choose T from the condition $R'^2(T) = p_*/2$ in problems with phase transitions. In this case, the limiting position of the phase-transition front $R_* = M(b)Z_*(b)$ is found from the energy integral (5.3), where

$$Z_*(b) = \left(1 + \frac{\nu + 1}{2e} Ap_*\right)^{-1/(\nu+1)} \tag{5.16}$$

The equation takes the following form in terms of b:

$$b = (E - B)F_\nu^{-1}(Z_*)G_\nu(Z_*). \tag{5.17}$$

The latter two equations, together with Eqs. (5.10) and the first relation in (5.4), are the system for determination of the parameters ξ, a, b, and Z_*.

In the case of a shock wave ($e = 0$), the situation is much simpler. It follows from the second equality in (5.4) that $b = 0$. Therefore, Eqs. (5.10) and the first relation in (5.4) are determined by the parameters ξ and a. According to the first equality in (5.4), the function $R(t)$ takes the form (3.1).

Example 1. Let $e = 0$. We choose the families $V(x, \xi)$ and $P(x, \xi)$ entering the definition of the approximate solution of problem II in the form

$$V = \xi_1, \quad P = \xi_2 x^{(\nu+1)(\xi_1 - 1)} \qquad (\xi_1 > 0,\ \xi_2 > 0). \tag{5.18}$$

Equations (5.10) and the first relation in (5.4) produce the system for determination of the parameters ξ_1, ξ_2, and a:

$$a\xi_1^3 = (\nu + 3)\xi_2, \quad 2\xi_2 = a\xi_1^2, \quad \frac{a\xi_1^2}{2(\nu + 3)} + \frac{\xi_2}{\xi_1(\nu + 1)} = E.$$

We find that

$$\xi_1 = \frac{\nu + 3}{2}, \quad \xi_2 = (\nu + 1)E, \quad a = \frac{8(\nu + 1)}{(\nu + 3)^2} E. \tag{5.19}$$

With allowance for (3.1), we obtain

$$R(t) = (2(\nu + 1)E)^{1/(\nu+3)} t^{2/(\nu+3)}.$$

Remark 7. The approximate solution that we found coincides with the self-modeling solution (3.6) for $\nu = 1$. For $\nu = 0$, we have

$$R(t) = 2^{1/3} E^{1/3} t^{2/3} \simeq 1.25992 E^{1/3} t^{2/3},$$

which is in agreement with formula (3.9). The question of the existence of an exact solution in the case of spherical symmetry and of the closeness of the approximate and exact solutions is omitted here.

Example 2. Let $e \neq 0$ and Eq. (5.18) be the family of functions which determine the solution of problem II. We suppose that $e/p_* \ll 1$. It follows from system (5.16) and (5.17) that $Z_* \ll 1$ and $\lambda = b/a \ll 1$. According to what was said above, the parameter a is calculated by formula (5.19). Therefore, for the limiting position of the phase-transition front the following approximate formula holds:

$$R_* = (16(\nu + 1)E/(\nu + 3)^2 p_*)^{1/(\nu+1)}.$$

We have ignored the energy expenditures in phase transitions. One can make allowance for them in a relatively simple way if the approximate solution is found from Eq. (5.17):

$$\lambda = \frac{2(\nu + 3)}{(\nu + 1)(3\nu + 5)} \frac{e}{p_*}.$$

Together with (5.2), Eqs. (5.10), (5.4), and (5.16) define the functions ξ_1, ξ_2, a, A, and B of variables λ and E. The formula for the limiting value of R_* is fairly cumbersome, and we omit it here.

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